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The Painlevé approach for perturbed nonlinear equations: Bäcklund and Miura-type transformations

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Abstract. A modification of the singular manifold or WTC method is proposed for perturbed nonlinear PDEs and its application is illustrated with several examples. As a result, approximate auto-Bäcklund and Miura-type transformations are constructed for the equations under consideration.

1. Introduction

The singular manifold or WTC method for nonlinear PDEs was proposed in [1] and has already been used with success for many such equations (see, for instance, [2] and references therein). This method allows one to carry out singular analysis of PDEs. It is, however, more significant that in a number of cases we can truncate the related series and use them to obtain Lax pairs, auto-Bäcklund transformations, etc. In principle, this approach can be applied to equations with a small parameter as well. However, when this is used to find corrections to solutions of the reduced equation, a number of difficulties emerge. First, since perturbed and non-perturbed versions may have varying leading terms, their related singular expansions may also differ. Moreover, if these terms involve a small factor, its inverse powers appear in the expansions. The purpose of the present paper is to extend the original method and propose its simple modification for the above-mentioned cases, using the ideas of the alternative technique, namely Painlevé's α-test. This approach was originally proposed by Painlevé [3] for singular analysis of ODEs and arose as a generalization of the method of small parameters. In recent years its application for PDEs has received wide acceptance [4–7].

The essence of the α-method is as follows. Suppose we have a one-parameter ODE that depends analytically on a complex parameter α in a domain containing $\alpha = 0$. Consistently testing the terms in the Maclaurin expansion

$$u(x) = u_0(x) + \alpha u_1(x) + \cdots$$

for its general solution, a set of necessary conditions for the absence of movable branch points in the full solution can be found.

The problem is that, in practice, for non-perturbed equations the parameter α is artificially introduced by a suitable scaling in such a way that the limit $\alpha \to 0$ introduces major simplification to the equation of interest.

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The main idea of the paper is to combine the truncated WTC and α-Painlevé methods and apply them to PDEs with a small parameter.

Let us take a perturbed PDE

$$\sum_{i=0}^{m} \varepsilon^{i} E_i[u] = 0 \qquad m \in N \qquad |\varepsilon| \ll 1 \tag{1.1}$$

where $E_i[\ ]$ are some differential operators. Inserting an asymptotic series

$$u = u_0(x,t) + \varepsilon u_1(x,t) + \cdots \tag{1.2}$$

into (1.1) and equating the expressions of each order in ε to zero, one gets a set of equations for u_i:

$$E_0[u_0] = 0 \tag{1.3}$$

$$EL[u_1] = -E_1[u_0] \tag{1.4}$$

$$\vdots$$

with $EL[\ ]$ being the linearization of $E_0[\ ]$.

As in [1,2], assume initially that u_i in (1.2)–(1.4) can be expanded in a generalized Laurent series:

$$u_i = \sum_{j=-p_i}^{+\infty} w_{ij}(x,t) F^j(x,t) \qquad p_i \in N. \tag{1.5}$$

This is necessary to determine the possible resonances in singular expansions for u_i and then to verify whether they should also contain powers of logarithms and iterated logarithms, because perturbations of integrable PDEs, in general, may possess solutions which cannot be represented by Laurent series. If so, the above terms must be added, changing (1.5) into a multiple series, the simplest case of which is the double series

$$u_i = \sum_{j=-p_i}^{+\infty} \sum_{k=0}^{+\infty} w_{ijk} F^j (\log F)^k \qquad P_i \in N.$$

Such singular analysis for (1.3) is standard and well known [1,2], whereas some remarks should be made for equations (1.4), i.e. linear inhomogeneous equations.

First of all, there are two kinds of leading terms in such equations. In the first the leading terms on the RHS only depend on u_j ($j < i$). The second type is associated with the related homogeneous equation and defined as the linearization of the leading terms of (1.3) on the background of the dominant addend $w_{0,-p_0} F^{-p_0}$ in (1.5) for u_0. Requiring that all these terms balance determines the dominant behaviour of u_i ($i > 0$) in the neighbourhood of $F = 0$, i.e. the possible values of p_i and $w_{i,-p_i}$.

The following properties are immediately associated with the homogeneous form of (1.4) and can rigorously be justified.

Proposition 1. Let r_k be the resonances in expansion (1.5) related to the lowest-order equation (1.3). Then arbitrary coefficients in (1.5) for u_i ($i > 0$) may arise only at the powers $F^{-p_0+r_k}$, i.e. the same powers of F as for u_0 and in addition F^{-p_0-1} (one resonance is always -1).

Proposition 2. Free coefficients in (1.5) for $u_i (i > 0)$ can be set equal to zero without loss of generality. *In other words, they can be taken into account in the leading-order* approximation u_0.

Finally, it should be noted that a perturbed equation can also admit movable essential singularities (branched or unbranched) or even movable natural barriers, which are not recognized by local analysis. Therefore, it may be that the expansions obtained are limited to some class of solutions. In particular, when $p_i = p_0+1$ there always exists a compatibility condition for F caused by the right-hand side of (1.4), and such a limitation is obvious. This fact may, for example, indicate the existence of a class of solutions with essential singularities.

As is known, in many cases generalized singular expansion can be truncated [1, 2], but this imposes a restriction on the type of F. Substituting such finite expansions into the equations under study, one finds not only recursion relations for their coefficients, but additional constraints for F (the so-called singular manifold equations) as well. An analogous procedure is readily introduced for the expansions of u_i. For the singular manifold equations to be obtained in these cases, it is necessary to find the required number of coefficients via the system (1.3)–(1.4) and insert (1.2) into the full equation (1.1). Then, again separating powers of F and retaining terms of appropriate order of ε, one obtains the above-mentioned relations for F.

It was shown in [8] that the function V (or $\chi = V^{-1}$)

$$V = \frac{\partial}{\partial x} \log\left(F/\sqrt{F_x}\right) \tag{1.6}$$

is the best expansion function for singular analysis from the standpoint of invariance under the Möbius group. (The Painlevé property together with some details of the analysis are invariant under this group.) In so doing, V satisfies the Riccati equations:

$$V_x = -V^2 - S/2 \tag{1.7}$$

$$V_t = CV^2 - C_x V + (CS + C_{xx})/2 \tag{1.8}$$

with the compatibility condition

$$S_t + C_{xxx} + 2C_x S + CS_x = 0 \tag{1.9}$$

where we have used the following compact notation [8]:

$$C = -F_t/F_x$$
$$S = F_{xxx}/F_x - \tfrac{3}{2}F_{xx}^2/F_x^2.$$

As was also shown in [8] (section 4, remark 1), application of this function naturally introduces and explains truncation on the constant level, i.e. the use of the truncated series

$$u = \sum_{j=-p}^{0} w_j F^j \qquad \text{or} \qquad u = \sum_{j=0}^{p} W_j V^j \qquad p \in N$$

whenever PDEs are polynomial in dependent variables and their derivatives.

The use of such a series is highly important for perturbed nonlinear PDEs, because they allow one to derive approximate auto-Bäcklund transformations, Lax pairs, etc (e.g. by applying the technique described in [9]) correctly. The problem is that the special type of functional series underlying the WTC approach is constructed for arbitrary function $F(x,t)$, although the necessary balance of terms holds at every value of independent variables [1,2]. The singular analysis considers these infinite series near the singular hypersurface $F=0$, whereas Weiss *et al* [1] have shown that their truncated versions are simple sums and valid everywhere over the region. For these reasons in the cases of unperturbed PDEs they result in the above-mentioned transformations, and these transformations are valid not only on the singular hypersurface but at every value of the independent variables. In the case of perturbed PDEs we have the same situation, and expansions (1.2) with u_i in the form of sums remain correct, possibly outside some neighbourhood of the singular hypersurface. (This situation is usual for perturbation techniques.) As mentioned earlier, this neighbourhood is not essential for constructing the above functional series and is of no interest for the soliton PDEs from a physical point of view in most cases, although some classes of solutions (e.g. rational solutions or the so-called algebraic solitons) cannot be investigated in this way. Unfortunately, in the general case it is impossible to estimate such a neighbourhood.

Below we will confine ourselves to the asymptotic series (1.2) up to ε and use the above expansions. In view of (1.6)–(1.8) they take the form

$$u_i = \sum_{j=0}^{p_i} W_{ij}(x,t)V^j \tag{1.10}$$

and the singular manifold equations are expressed only in terms of S and C. It is also important to note that for all of the PDEs considered below the related series are truncated before the resonances, i.e. before the points where the logarithmic functions of F could arise [1,2]. So the use of simple sums (1.10) is indeed justified.

2. The KdV, MKdV and Kaup–Kuperschmidt equations with perturbations

2.1. Perturbed KdV

We start with the perturbed KdV equation (PKdV) of the following form

$$u_t + 6uu_x + u_{xxx} + \varepsilon(\alpha u^2 u_x + \gamma u u_{xxx} + \beta u_x u_{xx} + \delta u_{xxxxx}) = 0 \tag{2.1}$$

the particular case of an equation proposed in [10] for waves in a rotation flow.

For the coefficients of (1.10) for u_0 and u_1 in (1.2) to be determined, one needs to consider the first two equations of (1.3)–(1.4), namely

$$\frac{\partial}{\partial t}u_0 + 6u_0\frac{\partial}{\partial x}u_0 + \frac{\partial^3}{\partial x^3}u_0 = 0 \tag{2.2}$$

$$\frac{\partial}{\partial t}u_1 + 6\frac{\partial}{\partial x}(u_1u_0) + \frac{\partial^3}{\partial x^3}u_1 = -\left(\alpha u_0^2\frac{\partial}{\partial x}u_0 + \gamma u_0\frac{\partial^3}{\partial x^3}u_0 + \beta\frac{\partial}{\partial x}u_0\frac{\partial^2}{\partial x^2}u_0 + \delta\frac{\partial^5}{\partial x^5}u_0\right). \tag{2.3}$$

The unperturbed KdV (2.2) is well known [1,2] to have the resonances $r_1=-1$, $r_2=4$, $r_3=6$, and the corresponding expansion (1.10) is of the form

$$u_0 = -2V^2 + (C-4S)/6.$$

For this u_0 the balance between the leading terms of (2.3) (all the addends without $\partial u_1/\partial t$) determines the value of p_1 in the expansion for u_1, i.e. $p_1 = 4$. Thereafter substituting the latter with $W_{13} = 0$ (see proposition 2) into (2.3) and separating the powers of V, one has the following recursion relations:

$$\begin{aligned}
&V^7\colon\ -3W_{14} + \alpha - 3\beta + 90\delta - 6\gamma = 0\\
&V^6\colon\ W_{14,x} = 0\\
&V^5\colon\ 2C(-\alpha + 3\gamma) + 2S(-27W_{14} + 7\alpha - 21\beta + 630\delta - 42\gamma)\\
&\qquad + 9(-W_{14,xx} + 2W_{12}) = 0\\
&V^4\colon\ 3CW_{14,x} + 132SW_{14,x} + 2S_x(27W_{14} - 4\alpha + 24\beta - 630\delta + 30\gamma)\\
&\qquad + C_x(-9W_{14} + 2\alpha - 6\beta) + 3(W_{14,t} + W_{14,xxx} + 6W_{12,x} + 30W_{11}) = 0\\
&V^3\colon\ C^2\alpha + 20CS(-\alpha + 3\gamma) + 2S^2(-135W_{14} + 32\alpha - 90\beta + 2772\delta - 192\gamma)\\
&\qquad + 54S(-W_{14,xx} + 2W_{12}) - 54S_xW_{14,x} + 6S_{xx}(-3W_{14} - 4\beta + 126\delta - 6\gamma)\\
&\qquad + 6C_{xx}(3W_{14} + \beta) + 54(-W_{12,xx} - W_{11,x} + 4W_{10}) = 0.
\end{aligned}\tag{2.4}$$

Solving them sequentially, one identifies the terms order by order in powers of V, and u_1 becomes of the form:

$$\begin{aligned}
u_1 = {}& (\alpha - 3\beta + 90\delta - 6\gamma)V^4/3 + \left[C(\alpha - 3\gamma) + 2S(\alpha - 3\beta + 90\delta - 6\gamma)\right]V^2/9\\
&+ \left[2S_x(-7\alpha + 9\beta - 360\delta + 36\gamma) + C_x(-\alpha - 3\beta + 270\delta - 12\gamma)\right]V/90\\
&+ \left[-5C^2\alpha + 40CS(\alpha - 3\gamma) + 10S^2(\alpha - 9\beta + 198\delta - 6\gamma)\right.\\
&\left.+ 48S_{xx}(\alpha - 2\beta + 45\delta - 3\gamma) + 3C_{xx}(-\alpha + 17\beta - 630\delta + 18\gamma)\right]/1080
\end{aligned}$$

with the condition (2.4) associated with the resonance $r_1 = -1$ of the reduced equation satisfied identically.

Then inserting (1.2) with u_0 and u_1 thus defined into equation (2.1) and omitting all the terms of order $O(\varepsilon)$, one again sets the coefficients of like powers in V equal to zero in order to obtain the singular manifold equation. In so doing, the coefficient of V^2 results in the following relation:

$$\begin{aligned}
90(C_x - S_x) + \varepsilon\big[&5(\alpha - 3\gamma)C_t + (8\alpha - 21\beta + 540\delta - 39\gamma)C_{xxx} + 10(\alpha - 3\beta + 90\delta - 6\gamma)S_t\\
&+ 3(3\alpha - \beta + 150\delta - 24\gamma)S_xS + 15(\alpha - 2\beta + 60\delta - 6\gamma)S_xC\\
&+ 3(7\alpha - 19\beta + 510\delta - 36\gamma)SC_x + (7\alpha - 9\beta + 270\delta - 36\gamma)S_{xxx}\big] \cong 0.
\end{aligned}$$

Taking into account the linkage (1.9) between C and S, it can be simplified, and we obtain for C:

$$\begin{aligned}
&C \cong S + \lambda_1 + \varepsilon\left[\delta S_{xx} + (\gamma/4 - \delta)S^2 + \lambda_1\gamma S/6 + \lambda_2\right]\\
&\lambda_1, \lambda_2 = \text{constant}.
\end{aligned}\tag{2.5}$$

As a result the approximate expansion for u takes, in view of (2.5), the following form:

$$\begin{aligned}
u = {}& (-12V^2 - 3S + \lambda_1)/6 + \varepsilon\big[-72V^4(-\alpha + 6\gamma + 3\beta - 90\delta) - 72V^2S(-\alpha + 5\gamma + 2\beta - 60\delta)\\
&- 24V^2\lambda_1(-\alpha + 3\gamma) + 36VS_x(-\alpha + 4\gamma + \beta - 30\delta) - 9S^2(-\alpha + 3\gamma + 2\beta - 40\delta)\\
&- 6S\lambda_1(-\alpha + 3\gamma) - 9S_{xx}(-\alpha + 2\gamma + \beta - 10\delta) + 36\lambda_2 - \lambda_1^2\alpha\big]/216 + O(\varepsilon^2).
\end{aligned}\tag{2.6}$$

The singular manifold equation (2.5) is seen to depend on just two parameters of (2.1). This fact permits us to establish a linkage between equations of type (2.1) with the same δ, γ but differing α, β.

Keeping in mind (1.7), (2.6) can be presented in terms of V:

$$u \cong (-6V^2+6V_x+\lambda_1)/6+\varepsilon\big[36V^4(\alpha-20\delta+5\gamma)+72V^2V_x(-\alpha-20\delta-\gamma) + 12V^2\lambda_1(-\alpha-3\gamma)+36VV_{xx}(-\alpha-\beta+50\delta-6\gamma)+36V_x^2(-\beta+30\delta-\gamma) + 12V_x\lambda_1(\alpha+3\gamma)+18V_{xxx}(\alpha+\beta-10\delta+2\gamma)+\alpha\lambda_1^2+36\lambda_2\big]/216. \tag{2.7}$$

After that, the relationship sought is easily derived:

$$u-u' \cong \varepsilon/12\big[(\alpha'-\alpha)(u_{xx}+2u^2)-(\beta'-\beta)u_{xx}\big]. \tag{2.8}$$

Here u corresponds to (2.1) with some $\alpha, \beta, \gamma, \delta$, and u' to (2.1) with the same γ, δ but α', β'. By this means we have obtained the approximate transformation between the equations of type (2.1) with various parameters of the perturbation. In the case that one of them is integrable, this mapping is called canonical [11]. There are four such cases: KdV, higher-order KdV $(\alpha,\gamma,\beta,\delta)=\delta(30,10,20,1)$ [12], modified Sawada–Kotera $\delta(45,15,15,1)$, and modified Kaup–Kuperschmidt $\delta(45,15,75/2,1)$ equations. The latter two can be derived from the classical analogue by formally replacing $u \to u+\text{constant}$ [13].

As an example, the typical profiles of the first corrections corresponding to KdV and high-order KdV two-soliton collisions are depicted in figures 1 and 2. The results are adduced in a coordinated system moving with the velocity of one of the solitons under consideration.

In addition to the above, an approximate Bäcklund transformation is associated with (2.5)–(2.7). Indeed, keeping in mind (2.5) and replacing S by

$$S \cong 2q-\lambda_1/3+\varepsilon\big[24\lambda_1(\gamma-10\delta)q+\lambda_1^2(20\delta+\gamma)-36\lambda_2\big]/108$$

the relation (1.9) leads to the equation

$$q_t+6q_xq+q_{xxx}+\varepsilon\big[\delta q_{xxxxx}+5(\gamma-4\delta)q_xq^2+(3\gamma-10\delta)q_xq_{xx}+\gamma qq_{xxx}\big]+O(\varepsilon^2)=0. \tag{2.9}$$

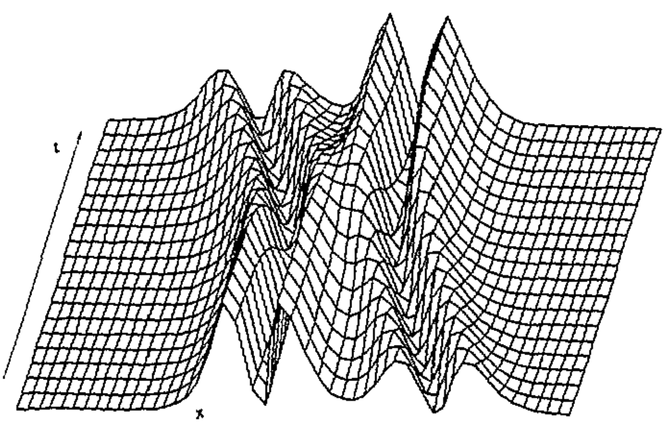

Figure 1. Plot of the first correction to the high-order KdV two-soliton solution (wavenumbers: $k_1=5$, $k_2=6$) for the PKdV equation at $\alpha=0$, $\beta=50$, $\gamma=10$, and $\delta=1$.

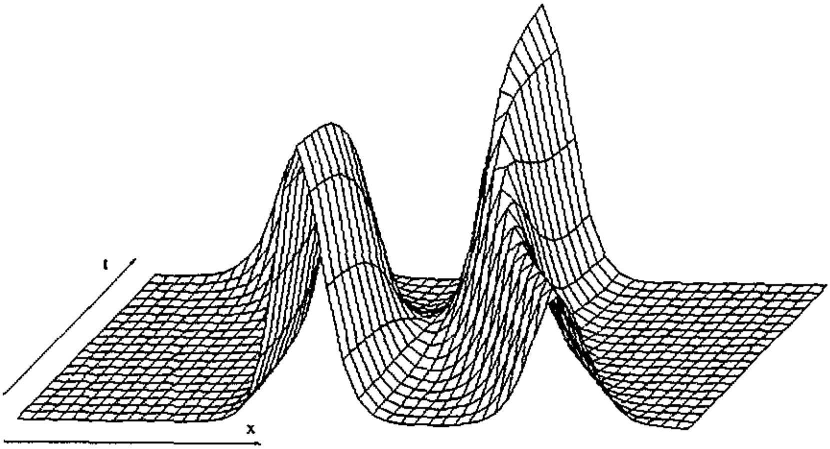

Figure 2. Plot of the first correction to the KdV two-soliton solution (wavenumbers: $k_1 = 5$, $k_2 = 6$) for the PKdV equation at $\alpha = 1$, $\beta = -1$, $\gamma = 0$, and $\delta = 0$.

First, it is of the type (2.1), and hence (2.5)–(2.7) determine the one-parameter (λ_2 can be included into λ_1 by $\lambda_1 \to \lambda_1 + \varepsilon\lambda_2$) Bäcklund transformation between (2.1) and (2.9). Moreover, for $\alpha = 5(\gamma - 4\delta)$; $\beta = 3\gamma - 10\delta$ these equations are identical, and we have the auto-transformation. Since the above procedures are also applicable to (2.9), in fact we have obtained the auto-transformation for any choice of the parameters.

In figure 3 the results obtained for equation (2.1) are schematically presented.

2.2. *Perturbed MKdV*

A perturbed MKdV equation arises, for instance, in the theory of quasi-one-dimensional solids [14] and in liquid-crystal hydrodynamics [15]. Here we will consider this equation with a perturbation of the most general type:

$$u_t + 6u^2u_x + u_{xxx} + \varepsilon(\alpha u_{xxxxx} + \beta u^2 u_{xxx} + \gamma u u_x u_{xx} + \delta u_x^3 - \zeta u u^4 u_x) = 0. \qquad (2.10)$$

The truncated series and resonances associated with its unperturbed version (1.3) are:

$$u_0 = \mathrm{i}V \qquad r_1 = -1 \qquad r_2 = 3 \qquad r_3 = 4.$$

The first-order equation of (1.4) respectively determines $p_1 = 3$ for u_1 and the form of the coefficients in (1.10):

$$\begin{aligned}
W_{13} &= -\mathrm{i}(120\alpha - 6\beta - \delta - 2\gamma - \zeta)/30 \\
W_{12} &= 0 \qquad \text{(see proposition 2)} \\
W_{11} &= -\mathrm{i}\big[(-120\alpha + 6\beta + \delta + 2\gamma + \zeta)C + (480\alpha - 44\beta + \delta - 8\gamma - 9\zeta)S\big]/120 \\
W_{10} &= -\mathrm{i}\big[S_x(-180\alpha + 14\beta + 4\delta + 3\gamma + 4\zeta) + C_x(120\alpha - 6\beta - \delta - 2\gamma - \zeta)\big]/120.
\end{aligned}$$

Further substitution of (1.2) with these u_0 and u_1 into (2.10) gives rise to the equations for S and C. For instance, from the coefficient of V^2 one has

$$\begin{aligned}
120(S - C) + \varepsilon\big[&3(120\alpha - 6\beta - \delta - 2\gamma - \zeta)C_{xx} + 3(-120\alpha + 8\beta - 7\delta + 6\gamma + 3\zeta)S_{xx} \\
&+ (-120\alpha + 6\beta + \delta + 2\gamma + \zeta)C^2 + 10(60\alpha - 5\beta - \gamma - \zeta)CS \\
&+ (-144\alpha + 16\beta - 5\delta + 4\gamma + 3\zeta)5S^2\big] \cong 0.
\end{aligned}$$

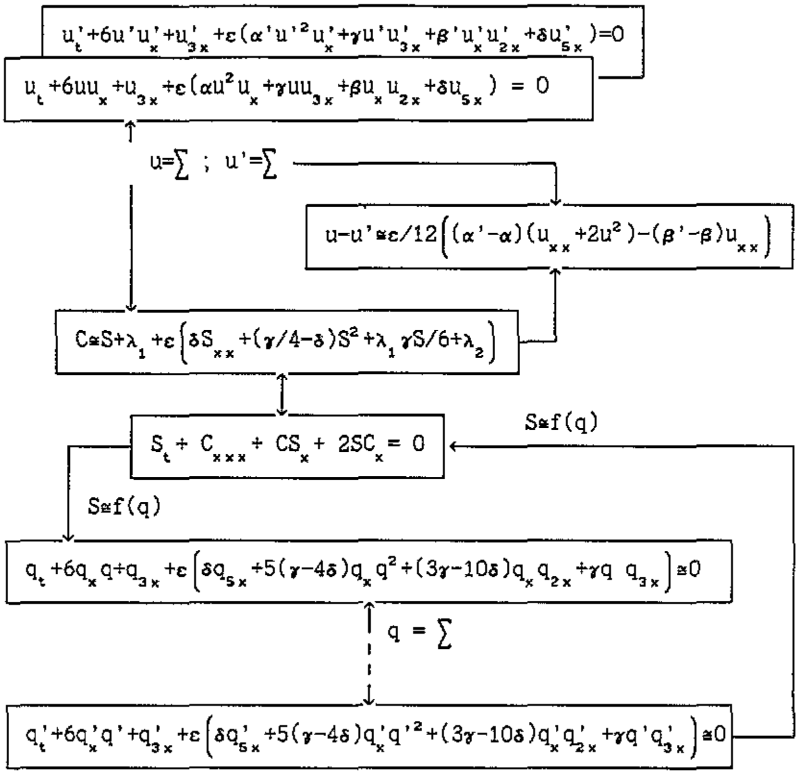

Figure 3. The properties of the perturbed KdV equation associated with its singular manifold equation.

Proceeding in the same manner as before, one gets

$$C \cong S + \varepsilon\big[\alpha S_{xx} + (\beta/4 - \alpha)S^2\big] + \varepsilon(-20\alpha + \beta - 4\delta + 2\gamma + \zeta)(S^2 + S_{xx})/20. \qquad (2.11)$$

As this takes place, the remaining coefficients are simplified and lead to the constraint $\zeta = 20\alpha - \beta - 2\gamma + 4\delta$. After that, (2.11) reduces to

$$C \cong S + \varepsilon\big[\alpha S_{xx} + (\beta/4 - \alpha)S^2\big] \qquad (2.12)$$

and we arrive at the net expression for $u(x,t)$:

$$u = \mathrm{i}V - \mathrm{i}\varepsilon\big\{4(20\alpha - \beta - \delta)V^3 + \big[-(20\alpha - \beta - \delta)S + (60\alpha - 7\beta + 2\gamma - 7\delta)S\big]V + (20\alpha - \beta - \delta)S_x + (-20\alpha + 2\beta - \gamma + 4\delta)S_x\big\}/24 + \mathrm{O}(\varepsilon^2). \qquad (2.13)$$

Again, the singular manifold equation (2.12) depends on just two parameters of (2.10), and the corresponding transformation between the equations of the type (2.10) with various γ, δ can easily be derived:

$$u' \cong u - \varepsilon\big[(-\gamma + 3\delta + \gamma' - 3\delta')u_{xx} + 2(-\gamma + 2\delta + \gamma' - 2\delta')u^3\big]/12. \qquad (2.14)$$

This mapping is canonical when u or u' corresponds to the integrable cases of (2.10): the MKdV and higher-order MKdV $(\alpha, \beta, \gamma, \delta, \zeta) = \alpha(1, 10, 40, 10, -30)$ [12] equations.

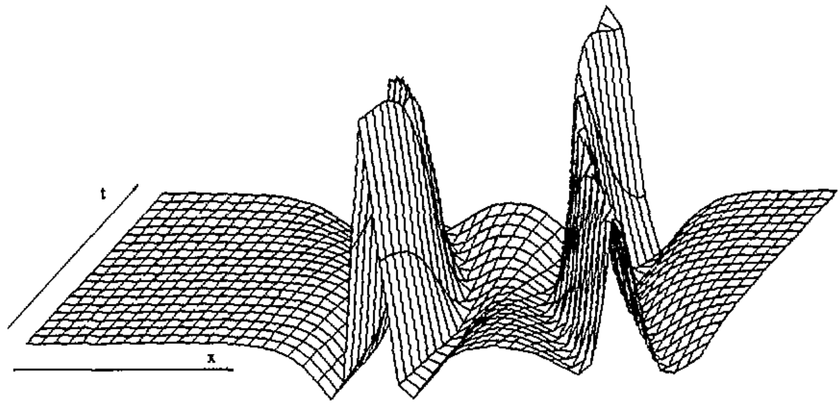

Figure 4. Plot of the first correction to the high-order MKdV two-soliton solution (wavenumbers: $k_1 = 1$, $k_2 = 1.2$) for equation (2.10) at $\alpha = -0.1$, $\beta = -1$, $\gamma = -2$, $\delta = 0$, and $\zeta = 3$.

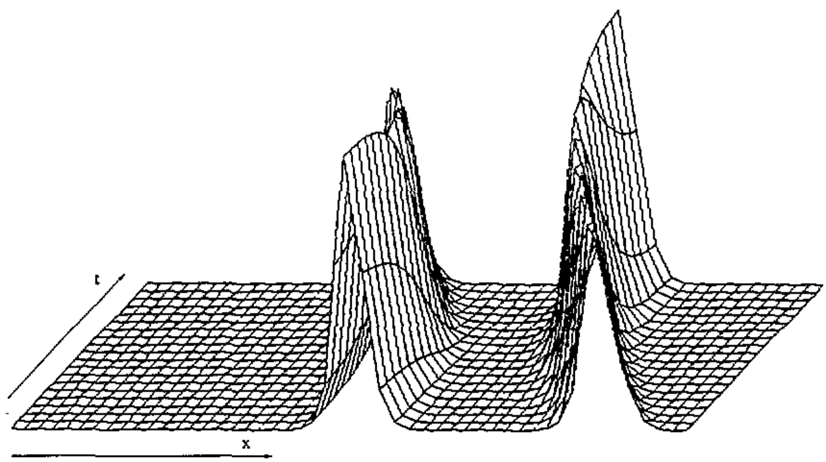

Figure 5. Plot of the first correction to the high-order MKdV two-soliton solution (wavenumbers: $k_1 = 1$, $k_2 = 1.1$) for equation (2.10) at $\alpha = 0.1$, $\beta = 1$, $\gamma = 1$, $\delta = 0$, and $\zeta = -1$.

Figures 4 and 5 demonstrate the typical profiles of the first corrections to high-order MKdV collisions.

It should be noted that the singular manifold equation (2.12) is of the same type as (2.5) for the perturbed KdV (2.1), and a similar auto-Bäcklund transformation also exists for (2.10). Moreover, this permits one to prolong a Miura transformation well known for the unperturbed KdV and MKdV [12] and find its approximate analogue.

Eliminating S from (2.13) via (1.7), one has

$$u \cong \mathrm{i}V + \varepsilon\mathrm{i}\big[2(-2\beta - 2\delta + \gamma)V^3 + 4(10\alpha - \beta)VV_x + (\beta + 3\delta - \gamma)V_{xx}\big]/12$$

whence it follows that, in turn, V can be expressed approximately in terms of u:

$$V \cong -\mathrm{i}u + \varepsilon\big[2\mathrm{i}(2\beta + 2\delta - \gamma)u^3 + 4(10\alpha - \beta)uu_x + \mathrm{i}(\beta + 3\delta - \gamma)u\big]/12.$$

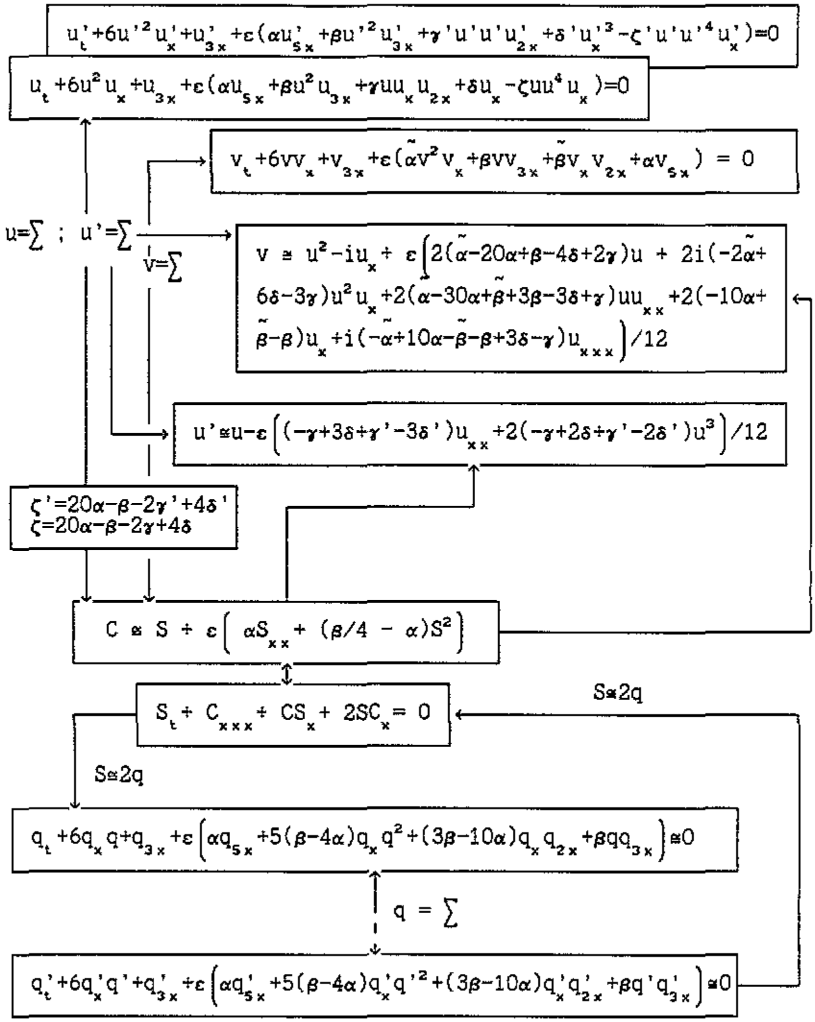

Figure 6. Properties of the perturbed MKdV equation.

Inserting the latter into (2.7), we arrive at the prolongation sought:

$$u_{\mathrm{PKdV}} = u^2 - \mathrm{i}u_x + \varepsilon\big[2(\alpha_{\mathrm{PKdV}} - 20\alpha + \beta - 4\delta + 2\gamma)u^4 + 2\mathrm{i}(-2\alpha_{\mathrm{PKdV}} + 6\delta - 3\gamma)u^2u_x$$
$$+ 2(\alpha_{\mathrm{PKdV}} - 30\alpha + \beta_{\mathrm{PKdV}} + 3\beta - 3\delta + \gamma)uu_{xx} + 2(-10\alpha + \beta_{\mathrm{PKdV}} - \beta)u_x^2$$
$$+ \mathrm{i}(-\alpha_{\mathrm{PKdV}} + 10\alpha - \beta_{\mathrm{PKdV}} - \beta + 3\delta - \gamma)u_{xxx}\big]/12. \qquad (2.15)$$

As is evident from the foregoing, the mappings (2.8), (2.14) and the prolonged Miura transformation (2.15) are of the same nature and associated with the related singular manifold equations. For this reason such mappings will be named Miura-type transformations.

The results obtained for (2.10) are presented schematically in figure 6.

In the next two examples we will only adduce the net results for the equations under consideration.

2.3. The perturbed Kaup–Kuperschmidt (KK) equation

The perturbed KK equation

$$u_t + 180u^2u_x + 75u_xu_{xx} + 30uu_{xxx} + u_{xxxxx} + \varepsilon(\alpha u^3u_x + \beta u^3 + \gamma uu_xu_{xx} + \delta u^2u_{xxx} + \sigma uu_{xxxxx} + \zeta u_xu_{xxxx} + \xi u_{xx}u_{xxx} + \mu u_{xxxxxxx}) = 0 \qquad (2.16)$$

as well as the perturbed KdV (2.1), is a particular case of the equation proposed in [10].

Proceeding as shown previously, the expressions for u and the singular manifold equation are derived:

$$C \cong S_{xx} + S^2/4 + \varepsilon\left[6(\sigma - 24\mu)SS_{xx} + (\sigma - 30\mu)S^3 + 72\mu S_{xxxx} + 6(\sigma - 33\mu)S_x^2\right]/72$$

$$u = -V^2/2 - S/6 + \varepsilon\big[12(7434\mu - 18\beta + 8\gamma - 339\sigma)(3V^4 + 2V^2S) + 6(-9450\mu + 22\beta - 10\gamma + 435\sigma)VS_x + 2(8442\mu - 22\beta + 10\gamma - 411\sigma)S_{xx} + (19\,782\mu - 50\beta + 22\gamma - 909\sigma)S^2\big]/8640 + O(\varepsilon^2)$$

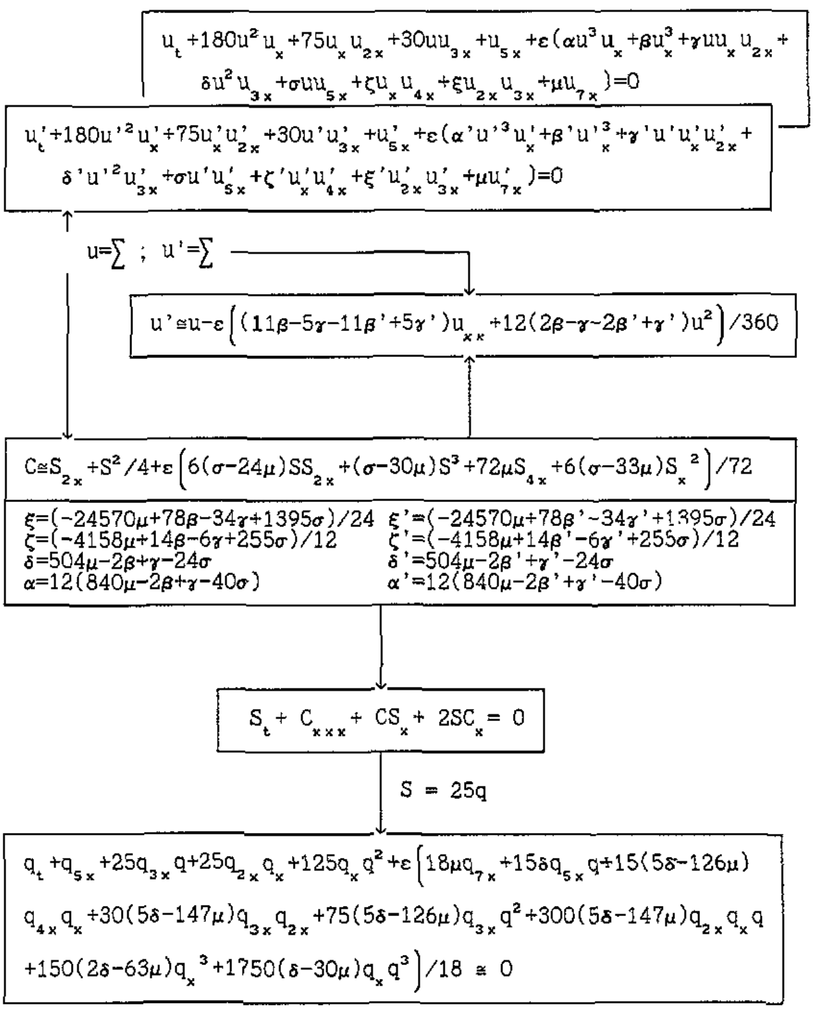

Figure 7. Properties of the perturbed KK equation.

provided that

$$\begin{aligned} \xi &= (-24\,570\mu + 78\beta - 34\gamma + 1395\sigma)/24 \\ \zeta &= (-4158\mu + 14\beta - 6\gamma + 255\sigma)/12 \\ \delta &= 504\mu - 2\beta + \gamma - 24\sigma \\ \alpha &= 12(840\mu - 2\beta + \gamma - 40\sigma) \end{aligned} \tag{2.17}$$

and the formulae analogous to (2.8), (2.14) are of the form

$$u' \cong u - \varepsilon\big[(11\beta - 5\gamma - 11\beta' + 5\gamma')u_{xx} + 12(2\beta - \gamma - 2\beta' + \gamma')u^2\big]/360. \tag{2.18}$$

The only integrable case of (2.16) corresponds to the KK hierarchy at

$$(\alpha, \beta, \gamma, \delta, \sigma, \zeta, \xi, \mu) = \mu(2016, 630, 2268, 504, 42, 147, 252, 1).$$

Note also that relation (1.9) results in the perturbed Sawada–Kotera equation [10] ($S = 25q$):

$$\begin{aligned} q_t + q_{xxxxx} &+ 25q_{xxx}q + 25q_{xx}q_x + 125q_xq^2 + \varepsilon\big[18\mu q_{xxxxxxx} + 15\delta q_{xxxxx}q \\ &+ 15(5\delta - 126\mu)q_{xxxx}q_x + 30(5\delta - 147\mu)q_{xxx}q_{xx} + 75(5\delta - 126\mu)q_{xxx}q^2 \\ &+ 300(5\delta - 147\mu)q_{xx}q_xq + 150(2\delta - 63\mu)q_x^3 \\ &+ 1750(\delta - 30\mu)q_xq^3\big]/18 + O(\varepsilon^2) = 0. \end{aligned}$$

The results obtained are presented schematically in figure 7.

2.4. *The regularized long-wave equation*

The regularized long-wave equation (RLW) [16]

$$u_t + 6uu_x + u_{xxx} + \varepsilon u_{xxt} = 0 \tag{2.19}$$

leads to the following results

$$\begin{aligned} u &\cong C/6 - 2/3S - 2V^2 + \varepsilon\big[C_{xx} - 2C_xV + 2C(S/3 + V^2)\big] \\ C &\cong \lambda + S - \varepsilon(2S_{xx} + \lambda^2 + 2\lambda S + S^2)/2. \end{aligned} \tag{2.20}$$

It turns out that the latter is identical to (2.5) with $\gamma = -6$, $\delta = -1$, $\lambda_1 = \lambda$, $\lambda_2 = -\lambda^2/2$; and this permits one to derive the Miura-type transformation between (2.1) and (2.19):

$$u_{\mathrm{RLW}} \cong u - \varepsilon\big[(-\alpha + \beta + 18)u_{xx} - 2\alpha u^2\big]/12.$$

In addition, the auto-Bäcklund transformation is associated with (2.20) as shown above.

These results are presented schematically in figure 8.

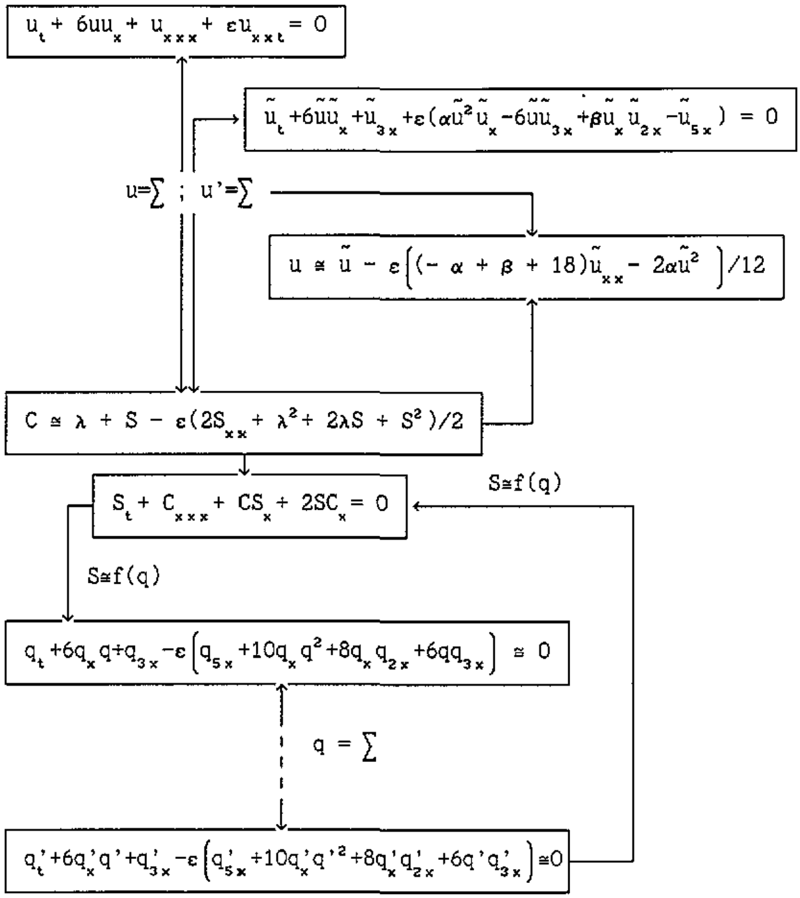

Figure 8. Properties of the RLW equation.

3. Conclusion

In the previous sections the combination of the α-Painlevé and singular manifold methods has been proposed for perturbed nonlinear PDEs as the generalization of the original WTC approach. The technique has been applied to several equations of interest for physical models.

Note that the results may be inapplicable to some special classes of solutions mentioned in the introduction. On the other hand, a number of them can be compared with ones obtained earlier. For examples, transformation (2.8) for the PKdV agrees with the results in [11], and the soliton solutions of the perturbed KdV and MKdV equations that can be constructed by transformations (2.8) and (2.14) are identical to those from [10, 17, 18]. In addition, one notes that all the transformations are easily verified by a direct substitution, and the approximate truncated expansions obtained for (2.1), (2.10) and (2.16) are identical to the exact ones when these equations belong to the KdV, MKdV or KK hierarchies [12].

I would like to point out a problem still to be studied. As noted in proposition 2, $u_i(i > 0)$ can partially be included into u_0. An inverse procedure is also important, because S and C may be modified when V is expanded in an asymptotic series. Application of the technique previously outlined for (1.7)–(1.8) leads to the following results:

$$S = S' - \varepsilon(w_{xx} + S'_x w + 2S' w_x) + O(\varepsilon^2)$$
$$C = C' + \varepsilon(w_x C' + w C'_x + w_t) + O(\varepsilon^2) \quad (3.1)$$
$$V = V' + \varepsilon\left[wV'^2 - w_x V' + (w_{xx} + S'w)/2\right] + O(\varepsilon^2)$$

(S', C' and V' also satisfy (1.7)–(1.8), and $w(x,t)$ is an arbitrary function). By this means the type of singular manifold equations can be modified and related transformations generalized. However, in practice it is not simple to use this fact and select a suitable function w, because in fact equation (3.1), which is also a Miura-type transformation, is also to be found.

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